THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 5

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1. Show that $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

2. Let $\{x_n\}$ be a sequence of real numbers. Define a new sequence of real numbers

$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

   for all $n \in \mathbb{N}$.

   (a) If $\lim_{n\to\infty} x_n = L$, show that $\lim_{n\to\infty} s_n = L$.

   (b) Is the converse of (a) true?

3. Let $\{y_n\}$ be a sequence such that $\lim_{n\to\infty} y_n = 2$. Using the $\varepsilon - N$ language, show that $\lim_{n\to\infty} \frac{y_n}{y_n^2 - 3} = 2$.

4. Find the limits of the following sequence defined by the recurrence relations:

$$x_1 := 1, x_{n+1} := \sqrt{2x_n}$$ (Monotone Convergence Theorem).

1. Show that $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

Recall: $x_n \to L$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$, s.t. $\forall n \geq N$,

$|x_n - L| < \varepsilon$.

So $x_n \not\to L$ means, $\exists \varepsilon_0 > 0$ s.t. for all $N \in \mathbb{N}$, there exists $n \geq N$

s.t. $|x_n - L| \geq \varepsilon_0$.

Pf: Let $L \in \mathbb{R}$ be given. Take $\varepsilon_0 = 1$, and let $N \in \mathbb{N}$ be given.

If $L \leq 0$, we can take $n = 2k$ for $k \in \mathbb{N}$ s.t. $k \geq \frac{N}{2}$.

Then $|x_n - L| = |x_{2k} - L| = |(-1)^{2k} - L| = |1 - L| \geq 1 = \varepsilon_0$

If $L > 0$, we can take $n = 2k+1$ for $k \geq \frac{N}{2}$. Then

$|x_n - L| = |x_{2k+1} - L| = |(-1)^{2k+1} - L| = |-1 - L|$

$= 1 + L > 1 = \varepsilon_0$.

2. Let $\{x_n\}$ be a sequence of real numbers. Define a new sequence of real numbers

$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for all $n \in \mathbb{N}$.

(a) If $\lim_{n\to\infty} x_n = L$, show that $\lim_{n\to\infty} s_n = L$.

(b) Is the converse of (a) true?

Pf: Since $x_n \to L$, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,
$|x_n - L| < \varepsilon$.

Now let $\varepsilon > 0$ be given.

$$|s_n - L| = \left| \frac{x_1 + x_2 + \cdots + x_n}{n} - L \right|$$

$$= \left| \frac{x_1 - L + x_2 - L + \cdots + x_n - L}{n} \right|$$

$$\leq \frac{1}{n}\left(|x_1 - L| + |x_2 - L| + \cdots + |x_n - L| \right)$$

Since $x_n \to L$, there is a $N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$,
$|x_n - L| < \frac{\varepsilon}{2}$.

$$= \frac{1}{n}\left(|x_1 - L| + |x_2 - L| + \cdots + |x_{N_1} - L| + |x_{N_1+1} - L| + \cdots + |x_n - L| \right)$$

So remains to deal with
$\frac{1}{n}\left(|x_1 - L| + \cdots + |x_{N_1} - L| \right)$. Since this sum is finite, let
$M = \max\{ |x_j - L| : j = 1, \ldots, N_1 \} < +\infty$.

Then, there is an $N_2 \in \mathbb{N}$ s.t. $N_2 > \frac{2N_1 M}{\varepsilon}$. Then for
$n \geq \max\{N_1, N_2\}$, we have

$$|s_n - L| \le \underbrace{\frac{1}{n}}_{< \frac{\varepsilon}{2N_1 M}} \Big(\underbrace{|x_1 - L| + \dots + |x_{N_1} - L|}_{N_1 \cdot M} \Big) + \frac{1}{n} \Big(\underbrace{|x_{N_1+1} - L|}_{< \frac{\varepsilon}{2}} + \dots + \underbrace{|x_n - L|}_{< \frac{\varepsilon}{2}} \Big)$$

$$< \frac{\varepsilon}{\cancel{2N_1M}} \cdot \cancel{N_1 M} + \underbrace{\frac{(n - N_1)}{n}}_{<1} \frac{\varepsilon}{2} < \varepsilon.$$

b) Converse is not true. Consider

$$x_n = \frac{1 + (-1)^n}{2} \qquad s_n = 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{4}, \frac{2}{5}, \dots \to \frac{1}{2}$$

$$= 0, 1, 0, 1, \dots$$

3. Let $\{y_n\}$ be a sequence such that $\lim_{n\to\infty} y_n = 2$. Using the $\varepsilon - N$ language, show that $\lim_{n\to\infty} \frac{y_n}{y_n^2-3} = 2$.

Pf: Let $\varepsilon > 0$. We estimate:

$$\left|\frac{y_n}{y_n^2-3} - 2\right| = \left|\frac{y_n - 2y_n^2 + 6}{y_n^2-3}\right| = \left|\frac{(y_n-2)(-2y_n-3)}{y_n^2-3}\right|$$

$$\leq |y_n-2|\left|\frac{-2y_n-3}{y_n^2-3}\right|$$

Since $y_n \to 2$, there is an $N_1 \in \mathbb{N}$ s.t. for all $n \geq N_1$, $y_n > 1$. Then $y_n^2 - 3 > -2 \Rightarrow \left|\frac{1}{y_n^2-3}\right| \leq \frac{1}{2}$.

and since y_n converges, y_n is bounded by some $M \in \mathbb{R}$. i.e. $|y_n| \leq M$.

So for $n \geq N_1$, $\left|\frac{-2y_n-3}{y_n^2-3}\right| \leq \left(\frac{2M+3}{2}\right)$ = constant not depending on ε or n

So $\left|\frac{y_n}{y_n^2-3} - 2\right| \leq |y_n - 2|\left(\frac{2M+3}{2}\right)$. Since $y_n \to 2$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$,

$$|y_n - 2| < \frac{2\varepsilon}{2M+3}$$

So taking $n \geq \max\{N_1, N_2\}$, we have

$$\left|\frac{y_n}{y_n^2-3} - 2\right| \leq |y_n-2|\left(\frac{2M+3}{2}\right) < \frac{2\varepsilon}{\cancel{2M+3}}\left(\frac{\cancel{2M+3}}{4}\right) = \varepsilon.$$

4. Find the limits of the following sequence defined by the recurrence relations:

$$x_1 := 1, x_{n+1} := \sqrt{2x_n}$$

Pf: Use MCT. Boundedness: $x_n \le 2$, and $x_n \le x_{n+1}$ by induction
and increasing

Base Case: $x_1 = 1 < 2$.

$x_1 = 1 < \sqrt{2} = x_2$. ✓

IH: Sps $x_k < 2$, $x_k \le x_{k+1}$ for some k.

$x_{k+1} = \sqrt{2x_k} < \sqrt{2 \cdot 2} = 2$.

$x_{k+2} = \sqrt{2x_{k+1}} \ge \sqrt{2x_k} = x_{k+1}$

So $\{x_n\}$ is bounded from above and increasing, hence by MCT, $\lim_{n\to\infty} x_n$ exists, call it $L \in \mathbb{R}$.

So taking limits as $n \to \infty$ in the recurrence relation, we have

$L = \sqrt{2L} \Rightarrow L(L-2) = 0 \Rightarrow L = \cancel{0}, 2$.

reject $L = 0$
since $x_n \ge x_1 = 1 > 0$